Matrix Element for $q\bar{q} \to t\bar{t}$ via V,A, and V$\pm$A currents

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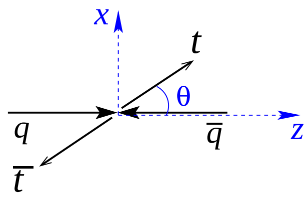
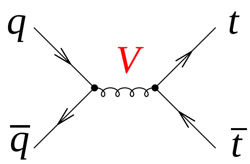
We define z-axis as the flight direction of q in $q\bar{q}$ rest frame (ZMF), and the z-x plane as the plane where $q\bar{q} \to t\bar{t}$ occures

Suppose θ as the production angle of top quark w.r.t. z-axis.



$q\bar{q} \to t\bar{t}$ via vector current



At $q\bar{q} \to t\bar{t}$ production, spin related part can be

$$\sigma \propto \sum_{q\bar{q}} |\mathfrak{M}_V|^2$$

$$\mathfrak{M}_V = (\bar{v}_{\bar{q}}\gamma^\mu u_q)(\bar{u}_t \gamma_\mu v_{\bar{t}}) = J^\mu_{q\bar{q}}(\lambda_q \lambda_{\bar{q}}) J_{t\bar{t},\mu}(\lambda'_t \lambda'_{\bar{t}})$$

where $J^\mu_{Q\bar{Q}} = \bar{v}_{\bar{Q}}\gamma^\mu u_Q$, $u_Q(\boldsymbol{p}_Q) = N_Q \begin{pmatrix} \chi_Q \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_Q}{E_Q + m_Q}\chi_Q \end{pmatrix}$ and $v_{\bar{Q}}(\boldsymbol{p}_{\bar{Q}}) = \begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_{\bar{Q}}}{E_{\bar{Q}} + m_{\bar{Q}}}\chi_{\bar{Q}} \\ \chi_{\bar{Q}} \end{pmatrix}$

Consider $J^\mu_{q\bar{q}}$ first, where

$$u_q(\boldsymbol{p}_q) = N_q \begin{pmatrix} \chi_q \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_q}{E_q + m_q}\chi_q \end{pmatrix} \qquad v_{\bar{q}}(\boldsymbol{p}_{\bar{q}}) = N_{\bar{q}} \begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_{\bar{q}}}{E_{\bar{q}} + m_{\bar{q}}}\chi_{\bar{q}} \\ \chi_{\bar{q}} \end{pmatrix} .$$

In $q\bar{q}\,(t\bar{t}\,)$ rest frame (ZMF),

$$\boldsymbol{p} \equiv \boldsymbol{p}_q = -\boldsymbol{p}_{\bar{q}} \qquad E \equiv E_q = E_{\bar{q}} \qquad m \equiv m_q = m_{\bar{q}} ,$$

then

$$\bar{v}_{\bar{q}}\gamma^\mu u_q = N_q N_{\bar{q}} \begin{pmatrix} \chi^\dagger_{\bar{q}} \frac{\boldsymbol{\sigma}\cdot(-\boldsymbol{p})}{E+m} & -\chi^\dagger_{\bar{q}} \end{pmatrix} (\gamma^0; \gamma^i) \begin{pmatrix} \chi_q \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi_q \end{pmatrix}$$

$$= -N_q N_{\bar{q}} \left(0; \begin{pmatrix} \chi^\dagger_{\bar{q}} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m} & \chi^\dagger_{\bar{q}} \end{pmatrix} \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} \begin{pmatrix} \chi_q \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi_q \end{pmatrix} \right)$$

$$= N_q N_{\bar{q}} \left(0; \chi^\dagger_{\bar{q}}\sigma_i \chi_q - \chi^\dagger_{\bar{q}} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\sigma_i \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi_q \right)$$

Suppose the case that $\boldsymbol{p} = (0, 0, p)$, and $m = 0$,

$$J^\mu_{q\bar{q}} = v_{\bar{q}}\gamma^\mu \bar{u}_q = N_q N_{\bar{q}} \left(0; \chi^\dagger_{\bar{q}} (\sigma_i - \sigma_3\,\sigma_i\,\sigma_3)\,\chi_q \right) = 2N_q N_{\bar{q}} \left(0; \chi^\dagger_{\bar{q}}\sigma_1 \chi_q, \chi^\dagger_{\bar{q}}\sigma_2\,\chi_q, 0 \right)$$

If we consider eigen states along z-axis for u_q and $v_{\bar{q}}$, i.e.

$$\chi_q(\uparrow)=\begin{pmatrix}1\\0\end{pmatrix},\ \chi_q(\downarrow)=\begin{pmatrix}0\\1\end{pmatrix},\ \chi_{\bar q}(\uparrow)=\begin{pmatrix}0\\1\end{pmatrix},\ \text{and}\ \chi_{\bar q}(\downarrow)=\begin{pmatrix}-1\\0\end{pmatrix}$$

then

$$J^\mu_{q\bar q}(\uparrow\uparrow)\propto\left\{0;\begin{pmatrix}0&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0&1\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix},0\right\}=(0;1,i,0)$$

$$J^\mu_{q\bar q}(\uparrow\downarrow)\propto\left\{0;\begin{pmatrix}-1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}-1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix},0\right\}=(0;0,0,0)$$

$$J^\mu_{q\bar q}(\downarrow\uparrow)\propto\left\{0;\begin{pmatrix}0&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0&1\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix},0\right\}=(0;0,0,0)$$

$$J^\mu_{q\bar q}(\downarrow\downarrow)\propto\left\{0;\begin{pmatrix}-1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix},0\right\}=(0;-1,i,0)$$

These results are consistent to helicity conservation. Remind that the polarization vectors for a spin-1 particle are

$$\varepsilon^\mu(\lambda=\pm)=(0;\mp1,-i,0)/\sqrt{2},\ \varepsilon^\mu(\lambda=0)=(0;0,0,1)\ .$$

This means there is no longitudinal component for $J^\mu_{q\bar q}$ and spin sum for the initial state becomes $\sum_{\lambda_q\lambda_{\bar q}=\uparrow\uparrow,\downarrow\downarrow}$.

Next, we consider $J_{t\bar t}$. In ZMF, i.e. $\boldsymbol{p}_t=-\boldsymbol{p}_{\bar t}$,

$$\begin{aligned}
J^\mu_{t\bar t}\ &=\bar u_t\gamma^\mu v_{\bar t}=N_tN_{\bar t}\begin{pmatrix}\chi_t\\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p_t}}{E_t+m_t}\chi_t\end{pmatrix}^\dagger\gamma^0\gamma^\mu\begin{pmatrix}-\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p_t}}{E_t+m_t}\chi_{\bar t}\\ \chi_{\bar t}\end{pmatrix}\\
&=N_tN_{\bar t}\begin{pmatrix}\chi_t^\dagger & -\chi_t^\dagger(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\end{pmatrix}\left\{\begin{pmatrix}1&0\\0&-1\end{pmatrix};\begin{pmatrix}0&\sigma_i\\-\sigma_i&0\end{pmatrix}\right\}\begin{pmatrix}-(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\chi_{\bar t}\\ \chi_{\bar t}\end{pmatrix},\\
&=N_tN_{\bar t}\big(-\chi_t^\dagger(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\chi_{\bar t}+\chi_t^\dagger(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\chi_{\bar t};-\chi_t^\dagger(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\sigma_i(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\chi_{\bar t}+\chi_t^\dagger\sigma_i\chi_{\bar t}\big)\\
&=N_tN_{\bar t}\big(0;\chi_t^\dagger\{\sigma_i-(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\,\sigma_i\,(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\}\chi_{\bar t}\big)
\end{aligned}$$

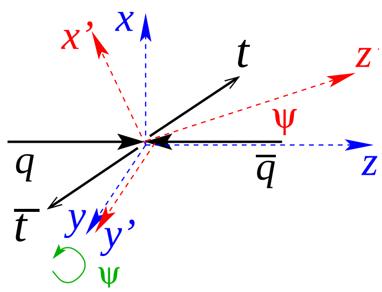
where

$$\boldsymbol{\alpha}=\frac{\boldsymbol{p}_t}{E_t+m_t}=k\begin{pmatrix}\sin\theta\\0\\\cos\theta\end{pmatrix},\ k^2=\left(\frac{p_t}{E_t+m_t}\right)^2=\frac{\gamma_t-1}{\gamma_t+1}$$

θ is the production angle of top quark w.r.t. z-axis. Then,

$$\begin{aligned}
J^\mu_{t\bar t}\ &\propto\big(0;\chi_t^\dagger\{\sigma_i-k^2(\sigma_1\sin\theta+\sigma_3\cos\theta)\sigma_i\,(\sigma_1\sin\theta+\sigma_3\cos\theta)\}\chi_{\bar t}\big)\\
&=\big(0;\chi_t^\dagger\{\sigma_i-k^2(\sigma_1\sigma_i\sigma_1\sin^2\theta+\sigma_3\sigma_i\sigma_1\sin\theta\cos\theta+\sigma_1\sigma_i\sigma_3\sin\theta\cos\theta+\sigma_3\sigma_i\sigma_3\cos^2\theta)\}\chi_{\bar t}\big)\\
&=\left(0;\chi_t^\dagger\begin{bmatrix}\sigma_1-k^2(\sigma_1\sin^2\theta+2\,\sigma_3\sin\theta\cos\theta-\sigma_1\cos^2\theta)\\ \sigma_2-k^2(-\sigma_2\sin^2\theta-\sigma_2\cos^2\theta)\\ \sigma_3-k^2(-\sigma_3\sin^2\theta+2\,\sigma_1\sin\theta\cos\theta+\sigma_3\cos^2\theta)\end{bmatrix}\chi_{\bar t}\right)\\
&=\left(0;\chi_t^\dagger\begin{bmatrix}(1+k^2\cos2\theta)\sigma_1-k^2\sin2\theta\cdot\sigma_3\\ (1+k^2)\sigma_2\\ (1-k^2\cos2\theta)\sigma_3-k^2\sin2\theta\cdot\sigma_1\end{bmatrix}\chi_{\bar t}\right)
\end{aligned}$$

Now we consider a new frame (top quantization frame) which is given by rotating the original frame by the angle ψ counterclockwise around y-axis.



2-component spinors for top and anti-top in a frame with the angle ψ w.r.t. z-axis i.e. eigen spinors along z'-axis are

$$\chi_{t\uparrow}=\begin{pmatrix}\cos\frac{\psi}{2}\\ \sin\frac{\psi}{2}\end{pmatrix},\ \chi_{t\downarrow}=\begin{pmatrix}-\sin\frac{\psi}{2}\\ \cos\frac{\psi}{2}\end{pmatrix}\text{ for top quark, and}$$

$$\chi_{\bar{t}\uparrow}=\begin{pmatrix}-\sin\frac{\psi}{2}\\ \cos\frac{\psi}{2}\end{pmatrix},\ \chi_{\bar{t}\downarrow}=\begin{pmatrix}-\cos\frac{\psi}{2}\\ -\sin\frac{\psi}{2}\end{pmatrix}\text{ for anti-top quark.}$$

In this case, the 4 eigen-states of $J^{\mu}_{t\bar{t}}$ for the quantization basis will become

$$J^{\mu}_{t\bar{t}}(\uparrow\uparrow)\propto\left(0;\begin{bmatrix}(1+k^2\cos2\theta)\cos\psi+k^2\sin2\theta\sin\psi\\ -i(1+k^2)\\ *\end{bmatrix}\right)$$

$$J^{\mu}_{t\bar{t}}(\uparrow\downarrow)=J^{\mu}_{t\bar{t}}(\downarrow\uparrow)\propto\left(0;\begin{bmatrix}-(1+k^2\cos2\theta)\sin\psi+k^2\sin2\theta\cos\psi\\ 0\\ *\end{bmatrix}\right)$$

$$J^{\mu}_{t\bar{t}}(\downarrow\downarrow)\propto\left(0;\begin{bmatrix}-(1+k^2\cos2\theta)\cos\psi-k^2\sin2\theta\sin\psi\\ -i(1+k^2)\\ *\end{bmatrix}\right)$$

Therefore

$$\begin{aligned}J_+ &\equiv J_{q\bar{q}}(\uparrow\uparrow)J_{t\bar{t}}(\uparrow\uparrow)=J_{q\bar{q}}(\downarrow\downarrow)J_{t\bar{t}}(\downarrow\downarrow)\\ &\propto(1+k^2\cos2\theta)\cos\psi+k^2\sin2\theta\,\sin\psi+(1+k^2)\\ &=1+k^2+\cos\psi+k^2\cos(2\theta-\psi)\end{aligned}$$

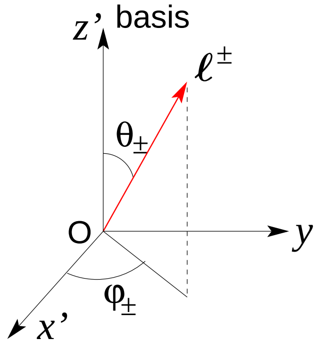
$$\begin{aligned}J_- &\equiv J_{q\bar{q}}(\uparrow\uparrow)J_{t\bar{t}}(\downarrow\downarrow)=J_{q\bar{q}}(\downarrow\downarrow)J_{t\bar{t}}(\uparrow\uparrow)\\ &\propto-(1+k^2\cos2\theta)\cos\psi-k^2\sin2\theta\,\sin\psi+(1+k^2)\\ &=1+k^2-\cos\psi-k^2\cos(2\theta-\psi)\end{aligned}$$

$$\begin{aligned}J_0 &\equiv J_{q\bar{q}}(\uparrow\uparrow)J_{t\bar{t}}(\uparrow\downarrow)=J_{q\bar{q}}(\uparrow\uparrow)J_{t\bar{t}}(\downarrow\uparrow)=-J_{q\bar{q}}(\downarrow\downarrow)J_{t\bar{t}}(\uparrow\downarrow)=-J_{q\bar{q}}(\downarrow\downarrow)J_{t\bar{t}}(\downarrow\uparrow)\\ &\propto-(1+k^2\cos2\theta)\sin\psi+k^2\sin2\theta\,\cos\psi\\ &=-\sin\psi+k^2\sin(2\theta-\psi)\end{aligned}$$

In the top quantization frame, using the lepton flight directions described by (θ_+,φ_+) for ℓ^+ and (θ_-,φ_-) for ℓ^- in a basis, the spinor of the top (anti-top) quark can be described as

$$u_t(\theta_+,\varphi_+)=e^{-i\varphi_+/2}\cos\frac{\theta_+}{2}u_{t\uparrow}+e^{i\varphi_+/2}\sin\frac{\theta_+}{2}u_{t\downarrow}$$

$$v_{\bar{t}}(\theta_-,\varphi_-)=e^{-i\varphi_-/2}\cos\frac{\theta_-}{2}v_{\bar{t}\downarrow}-e^{i\varphi_-/2}\sin\frac{\theta_-}{2}v_{\bar{t}\uparrow}$$



In this case,

$$
\begin{aligned}
J^{\mu}_{t\bar{t}} &= \left(e^{i\varphi_+/2}\cos\frac{\theta_+}{2}\bar{u}_{t\uparrow} + e^{-i\varphi_+/2}\sin\frac{\theta_+}{2}\bar{u}_{t\downarrow}\right)\gamma^{\mu}\left(-e^{i\varphi_-/2}\sin\frac{\theta_-}{2}v_{\bar{t}\uparrow} + e^{-i\varphi_-/2}\cos\frac{\theta_-}{2}v_{\bar{t}\downarrow}\right) \\
&= -e^{i(\varphi_+ + \varphi_-)/2}\cos\frac{\theta_+}{2}\sin\frac{\theta_-}{2}J_{t\bar{t}}(\uparrow\uparrow) + e^{i(\varphi_+ - \varphi_-)/2}\cos\frac{\theta_+}{2}\cos\frac{\theta_-}{2}J_{t\bar{t}}(\uparrow\downarrow) \\
&\quad -e^{-i(\varphi_+ - \varphi_-)/2}\sin\frac{\theta_+}{2}\sin\frac{\theta_-}{2}J_{t\bar{t}}(\downarrow\uparrow) + e^{-i(\varphi_+ + \varphi_-)/2}\sin\frac{\theta_+}{2}\cos\frac{\theta_-}{2}J_{t\bar{t}}(\downarrow\downarrow)
\end{aligned}
$$

$t\bar{t}$ spin correlations can be seen as angular correlations of decay products.

$$
\begin{aligned}
|J_{q\bar{q}}(\uparrow\uparrow)J_{t\bar{t}}|^2 &= \Big|-e^{i(\varphi_+ + \varphi_-)/2}\cos\frac{\theta_+}{2}\sin\frac{\theta_-}{2}J_{q\bar{q}}(\uparrow\uparrow)J_{t\bar{t}}(\uparrow\uparrow) \\
&\quad +e^{i(\varphi_+ - \varphi_-)/2}\cos\frac{\theta_+}{2}\cos\frac{\theta_-}{2}J_{q\bar{q}}(\uparrow\uparrow)J_{t\bar{t}}(\uparrow\downarrow) \\
&\quad -e^{-i(\varphi_+ - \varphi_-)/2}\sin\frac{\theta_+}{2}\sin\frac{\theta_-}{2}J_{q\bar{q}}(\uparrow\uparrow)J_{t\bar{t}}(\downarrow\uparrow) \\
&\quad +e^{-i(\varphi_+ + \varphi_-)/2}\sin\frac{\theta_+}{2}\cos\frac{\theta_-}{2}J_{q\bar{q}}(\uparrow\uparrow)J_{t\bar{t}}(\downarrow\downarrow)\Big|^2 \\
&= \Big|-e^{i(\varphi_+ + \varphi_-)/2}\cos\frac{\theta_+}{2}\sin\frac{\theta_-}{2}J_+ + e^{i(\varphi_+ - \varphi_-)/2}\cos\frac{\theta_+}{2}\cos\frac{\theta_-}{2}J_0 \\
&\quad -e^{-i(\varphi_+ - \varphi_-)/2}\sin\frac{\theta_+}{2}\sin\frac{\theta_-}{2}J_0 + e^{-i(\varphi_+ + \varphi_-)/2}\sin\frac{\theta_+}{2}\cos\frac{\theta_-}{2}J_-\Big|^2 \\
&= \cos^2\frac{\theta_+}{2}\sin^2\frac{\theta_-}{2}|J_+|^2 + \cos^2\frac{\theta_+}{2}\cos^2\frac{\theta_-}{2}|J_0|^2 \\
&\quad +\sin^2\frac{\theta_+}{2}\sin^2\frac{\theta_-}{2}|J_0|^2 + \sin^2\frac{\theta_+}{2}\cos^2\frac{\theta_-}{2}|J_-|^2 \\
&\quad -2\cos^2\frac{\theta_+}{2}\sin\frac{\theta_-}{2}\cos\frac{\theta_-}{2}\mathrm{Re}(e^{i\varphi_-}J_+J_0^*) \\
&\quad +2\cos\frac{\theta_+}{2}\sin\frac{\theta_+}{2}\sin^2\frac{\theta_-}{2}\mathrm{Re}(e^{i\varphi_+}J_+J_0^*) \\
&\quad -2\cos\frac{\theta_+}{2}\sin\frac{\theta_+}{2}\sin\frac{\theta_-}{2}\cos\frac{\theta_-}{2}\mathrm{Re}\big(e^{i(\varphi_+ + \varphi_-)}J_+J_-^*\big) \\
&\quad -2\cos(\varphi_+ - \varphi_-)\cos\frac{\theta_+}{2}\sin\frac{\theta_+}{2}\cos\frac{\theta_-}{2}\sin\frac{\theta_-}{2}|J_0|^2 \\
&\quad +2\cos\frac{\theta_+}{2}\sin\frac{\theta_+}{2}\cos^2\frac{\theta_-}{2}\mathrm{Re}(e^{i\varphi_+}J_-^*J_0) \\
&\quad -2\sin^2\frac{\theta_+}{2}\cos\frac{\theta_-}{2}\sin\frac{\theta_-}{2}\mathrm{Re}(e^{i\varphi_-}J_-^*J_0)
\end{aligned}
$$

For $J_{q\bar{q}}(\uparrow\uparrow)J_{t\bar{t}} \longrightarrow J_{q\bar{q}}(\downarrow\downarrow)J_{t\bar{t}}$, we just set $J_\pm \longrightarrow J_\mp$ and $J_0 \longrightarrow -J_0$.

$$
\begin{aligned}
4\sigma &\propto |J_{q\bar{q}}(\uparrow\uparrow)J_{t\bar{t}}|^2 + |J_{q\bar{q}}(\downarrow\downarrow)J_{t\bar{t}}|^2 \\
&= \left(\cos^2\frac{\theta_+}{2}\sin^2\frac{\theta_-}{2} + \sin^2\frac{\theta_+}{2}\cos^2\frac{\theta_-}{2}\right)(|J_+|^2 + |J_-|^2) \\
&\quad +2\left(\cos^2\frac{\theta_+}{2}\cos^2\frac{\theta_-}{2} + \sin^2\frac{\theta_+}{2}\sin^2\frac{\theta_-}{2}\right)|J_0|^2 \\
&\quad -2\cos^2\frac{\theta_+}{2}\sin\frac{\theta_-}{2}\cos\frac{\theta_-}{2}\mathrm{Re}(e^{i\varphi_-}J_+J_0^* - e^{i\varphi_-}J_-^*J_0) \\
&\quad +2\cos\frac{\theta_+}{2}\sin\frac{\theta_+}{2}\sin^2\frac{\theta_-}{2}\mathrm{Re}(e^{i\varphi_+}J_+J_0^* - e^{i\varphi_+}J_-^*J_0) \\
&\quad -4\cos\frac{\theta_+}{2}\sin\frac{\theta_+}{2}\cos\frac{\theta_-}{2}\sin\frac{\theta_-}{2}\mathrm{Re}\big(e^{i(\varphi_+ + \varphi_-)}J_+J_-^*\big) \\
&\quad -4\cos(\varphi_+ - \varphi_-)\cos\frac{\theta_+}{2}\sin\frac{\theta_+}{2}\cos\frac{\theta_-}{2}\sin\frac{\theta_-}{2}|J_0|^2 \\
&\quad +2\cos\frac{\theta_+}{2}\sin\frac{\theta_+}{2}\cos^2\frac{\theta_-}{2}\mathrm{Re}(e^{i\varphi_+}J_-^*J_0 - e^{i\varphi_+}J_+J_0^*) \\
&\quad -2\sin^2\frac{\theta_+}{2}\cos\frac{\theta_-}{2}\sin\frac{\theta_-}{2}\mathrm{Re}(e^{i\varphi_-}J_-^*J_0 - e^{i\varphi_-}J_+J_0^*)
\end{aligned}
$$

$$
\begin{aligned}
= \;& (1-\cos\theta_+\cos\theta_-)\frac{|J_+|^2+|J_-|^2}{2}+(1+\cos\theta_+\cos\theta_-)|J_0|^2 \\
& -\sin\theta_+\cos\theta_-\mathrm{Re}\{e^{i\varphi_+}(J_+J_0^*-J_-^*J_0)\} \\
& -\cos\theta_+\sin\theta_-\mathrm{Re}\{e^{i\varphi_-}(J_+J_0^*-J_-^*J_0)\} \\
& -\sin\theta_+\sin\theta_-\big\{\mathrm{Re}\big(e^{i(\varphi_++\varphi_-)}J_+J_-^*\big)+\cos(\varphi_+-\varphi_-)|J_0|^2\big\}
\end{aligned}
$$

Here, if we choose ψ (**offdiagonal basis**) which satisfies

$$\sin\psi = k^2\sin(2\theta-\psi)$$

i.e.

$$\sin\psi = k^2(\sin2\theta\cos\psi - \cos2\theta\sin\psi)$$

$$\tan\psi = \frac{k^2\sin2\theta}{1+k^2\cos2\theta} = \frac{2(\gamma-1)\sin\theta\cos\theta}{\gamma+1+(\gamma-1)(1-2\sin^2\theta)} = \frac{(\gamma-1)\tan\theta}{\gamma+\tan^2\theta}$$

$$\frac{1}{\gamma}\tan\theta = \frac{\tan\theta-\tan\psi}{1+\tan\psi\tan\theta} = \tan(\theta-\psi)$$

In this frame,

$$J_+ \propto 1+k^2+\cos\psi+k^2\cos(2\theta-\psi)$$

$$J_- \propto 1+k^2-\cos\psi-k^2\cos(2\theta-\psi)$$

$$J_0 = 0$$

$$4\sigma \propto (1-\cos\theta_+\cos\theta_-)\frac{J_+^2+J_-^2}{2} - \sin\theta_+\sin\theta_-\cos(\varphi_++\varphi_-)J_+J_-$$

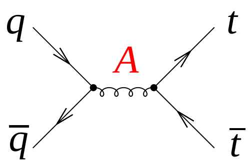
$$
\begin{aligned}
J_+J_- &\propto (1+k^2)^2 - \{\cos\psi+k^2\cos(2\theta-\psi)\}^2 \\
&= \sin^2\psi + 2k^2 + k^4\sin^2(2\theta-\psi) - 2k^2\cos\psi\cos(2\theta-\psi) \\
&= 2k^2\sin\psi\sin(2\theta-\psi) + 2k^2 - 2k^2\cos\psi\cos(2\theta-\psi) \\
&= 2k^2 - 2k^2\cos2\theta = 4k^2\sin^2\theta
\end{aligned}
$$

$$\frac{J_+^2+J_-^2}{2} = \frac{1}{2}(J_++J_-)^2 - J_+J_- \propto 2(1+k^2)^2 - 4k^2\sin^2\theta = 2(1+k^4)+4k^2\cos^2\theta$$

Top production angle distribution is corresponding to $\frac{J_+^2+J_-^2}{2}$. If top quark is ultra-relastivistic $k\to1$, then

$$\frac{J_+^2+J_-^2}{2} \to 4+4\cos^2\theta$$

$q\bar{q}\to t\bar{t}$ via axial-vector current



$$\mathfrak{M}_A = (\bar{v}_{\bar{q}}\gamma^\mu\gamma^5u_q)(\bar{u}_t\gamma_\mu\gamma^5v_{\bar{t}}) = A_{q\bar{q}}(\lambda_q\lambda_{\bar{q}})A_{t\bar{t}}(\lambda_t'\lambda_{\bar{t}}')$$

where $A^\mu_{Q\bar{Q}} = \bar{v}_{\bar{Q}}\gamma^\mu\gamma^5u_Q$, $u_Q(\boldsymbol{p}_Q) = N_Q\begin{pmatrix}\chi_Q \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_Q}{E_Q+m_Q}\chi_Q\end{pmatrix}$ and $v_{\bar{Q}}(\boldsymbol{p}_{\bar{Q}}) = N_{\bar{Q}}\begin{pmatrix}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_{\bar{Q}}}{E_{\bar{Q}}+m_{\bar{Q}}}\chi_{\bar{Q}} \\ \chi_{\bar{Q}}\end{pmatrix}$.

Assume

$$\boldsymbol{p}\equiv\boldsymbol{p}_q = -\boldsymbol{p}_{\bar{q}} = (0,0,p)\ ,\ m_q = m_{\bar{q}} = 0\ ,\ \text{and}\ E = E_q = E_{\bar{q}} = p,$$

then

$$
\begin{aligned}
A^{\mu}_{q\bar{q}} &= \bar{v}_{\bar{q}}\gamma^{\mu}\gamma^5 u_q \propto \begin{pmatrix} -\sigma_3\chi_{\bar{q}} \\ \chi_{\bar{q}} \end{pmatrix}^{\dagger} \gamma^0\gamma^{\mu}\gamma^5 \begin{pmatrix} \chi_q \\ \sigma_3\chi_q \end{pmatrix} \\
&= \begin{pmatrix} -\chi^{\dagger}_{\bar{q}}\sigma_3 & -\chi^{\dagger}_{\bar{q}} \end{pmatrix} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \right\} \begin{pmatrix} \sigma_3\chi_q \\ \chi_q \end{pmatrix} \\
&= \left\{ 0; \begin{pmatrix} \chi^{\dagger}_{\bar{q}}\sigma_i & -\chi^{\dagger}_{\bar{q}}\sigma_3\sigma_i \end{pmatrix} \begin{pmatrix} \sigma_3\chi_q \\ \chi_q \end{pmatrix} \right\} \\
&= \left\{ 0; \chi^{\dagger}_{\bar{q}}(\sigma_i\sigma_3 - \sigma_3\sigma_i)\chi_q \right\} \\
&= 2\left\{ 0; \chi^{\dagger}_{\bar{q}}(-i\sigma_2)\chi_q, \chi^{\dagger}_{\bar{q}}(i\sigma_1)\,\chi_q, 0 \right\}
\end{aligned}
$$

If we consider eigen states along z-axis for u_q and $v_{\bar{q}}$, i.e.

$$
\chi_q(\uparrow) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \chi_q(\downarrow) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \chi_{\bar{q}}(\uparrow) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \text{and}\ \chi_{\bar{q}}(\downarrow) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}
$$

then

$$
\begin{aligned}
A^{\mu}_{q\bar{q}}(\uparrow\uparrow) &\propto \left\{ 0; \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0 \right\} = (0; 1, i, 0) \\
A^{\mu}_{q\bar{q}}(\uparrow\downarrow) &\propto \left\{ 0; \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0 \right\} = (0; 0, 0, 0) \\
A^{\mu}_{q\bar{q}}(\downarrow\uparrow) &\propto \left\{ 0; \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0 \right\} = (0; 0, 0, 0) \\
A^{\mu}_{q\bar{q}}(\downarrow\downarrow) &\propto \left\{ 0; \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0 \right\} = (0; 1, -i, 0)
\end{aligned}
$$

The axial vector current gives helicity conservation as well.

Next, we consider $A_{t\bar{t}}$. In ZMF, i.e. $\boldsymbol{p_t} = -\boldsymbol{p}_{\bar{t}}$, $m_t = m_{\bar{t}}$, and $E_t = E_{\bar{t}}$

$$
\begin{aligned}
A^{\mu}_{t\bar{t}} &= \bar{u}_t\gamma^{\mu}\gamma^5 v_{\bar{t}} \propto \begin{pmatrix} \chi_t \\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p_t}}{E_t + m_t}\chi_t \end{pmatrix}^{\dagger} \gamma^0\gamma^{\mu}\gamma^5 \begin{pmatrix} -\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p_t}}{E_t + m_t}\chi_{\bar{t}} \\ \chi_{\bar{t}} \end{pmatrix} \\
&= \begin{pmatrix} \chi^{\dagger}_t & -\chi^{\dagger}_t(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}) \end{pmatrix} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \right\} \begin{pmatrix} \chi_{\bar{t}} \\ -(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\chi_{\bar{t}} \end{pmatrix}, \\
&= \left\{ \chi^{\dagger}_t\chi_{\bar{t}} - \chi^{\dagger}_t(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})^2\chi_{\bar{t}}\,;\, \chi^{\dagger}_t(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\sigma_i\chi_{\bar{t}} - \chi^{\dagger}_t\sigma_i(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\chi_{\bar{t}} \right\} \\
&= \left((1-\boldsymbol{\alpha}^2)\chi^{\dagger}_t\chi_{\bar{t}}\,;\, \chi^{\dagger}_t\{(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\sigma_i - \sigma_i\,(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\}\chi_{\bar{t}} \right)
\end{aligned}
$$

where $\boldsymbol{\alpha} = \dfrac{\boldsymbol{p_t}}{E_t + m_t} = k\begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$, $k^2 = \left(\dfrac{p_t}{E_t + m_t}\right)^2 = \dfrac{\gamma_t - 1}{\gamma_t + 1}$, then

$$
\begin{aligned}
\bar{u}_t\gamma^{\mu}\gamma^5 v_{\bar{t}} &\propto \left(*; \chi^{\dagger}_t\{k(\sigma_1\sin\theta + \sigma_3\cos\theta)\sigma_i - k\sigma_i\,(\sigma_1\sin\theta + \sigma_3\cos\theta)\}\chi_{\bar{t}} \right) \\
&= \left(*; \chi^{\dagger}_t\{k(\sigma_1\sigma_i - \sigma_i\sigma_1)\sin\theta + k(\sigma_3\sigma_i - \sigma_i\sigma_3)\cos\theta\}\chi_{\bar{t}} \right) \\
&= \left(*; \chi^{\dagger}_t \begin{bmatrix} 2i\,k\sigma_2\cos\theta \\ 2i\,k\sigma_3\sin\theta - 2i\,k\sigma_1\cos\theta \\ -2i\,k\sigma_2\sin\theta \end{bmatrix} \chi_{\bar{t}} \right)
\end{aligned}
$$

Suppose

$$
\chi_{t\uparrow} = \begin{pmatrix} \cos\frac{\psi}{2} \\ \sin\frac{\psi}{2} \end{pmatrix},\ \chi_{t\downarrow} = \begin{pmatrix} -\sin\frac{\psi}{2} \\ \cos\frac{\psi}{2} \end{pmatrix}\ \text{for top quark, and}
$$

$$
\chi_{\bar{t}\uparrow} = \begin{pmatrix} -\sin\frac{\psi}{2} \\ \cos\frac{\psi}{2} \end{pmatrix},\ \chi_{\bar{t}\downarrow} = \begin{pmatrix} -\cos\frac{\psi}{2} \\ -\sin\frac{\psi}{2} \end{pmatrix}\ \text{for anti-top quark,}
$$

then, the axial vector currents on the 4 eigen-states of $A_{t\bar{t}}$ will be

$$A^{\mu}_{t\bar{t}}(\uparrow\uparrow) \propto \left(*; \begin{bmatrix} 2k\cos\theta \\ -2i\,k\sin\psi\sin\theta - 2i\,k\cos\psi\cos\theta \\ * \end{bmatrix} \right)$$

$$A^{\mu}_{t\bar{t}}(\uparrow\downarrow) = A^{\mu}_{t\bar{t}}(\downarrow\uparrow) \propto \left(*; \begin{bmatrix} 0 \\ -2i\,k\cos\psi\sin\theta + 2i\,k\sin\psi\cos\theta \\ * \end{bmatrix} \right)$$

$$A^{\mu}_{t\bar{t}}(\downarrow\downarrow) \propto \left(*; \begin{bmatrix} 2k\cos\theta \\ 2i\,k\sin\psi\sin\theta + 2i\,k\cos\psi\cos\theta \\ * \end{bmatrix} \right)$$

Therefore

$$\begin{aligned} A_+ &\equiv A_{q\bar{q}}(\uparrow\uparrow)A_{t\bar{t}}(\uparrow\uparrow) = \{A_{q\bar{q}}(\downarrow\downarrow)A_{t\bar{t}}(\downarrow\downarrow)\} \\ &\propto 2k\{(1+\cos\psi)\cos\theta + \sin\psi\sin\theta\} = 2k\{\cos\theta + \cos(\psi-\theta)\} \end{aligned}$$

$$\begin{aligned} A_- &\equiv A_{q\bar{q}}(\uparrow\uparrow)A_{t\bar{t}}(\downarrow\downarrow) = \{A_{q\bar{q}}(\downarrow\downarrow)A_{t\bar{t}}(\uparrow\uparrow)\} \\ &\propto 2k\{(1-\cos\psi)\cos\theta - \sin\psi\sin\theta\} = 2k\{\cos\theta - \cos(\psi-\theta)\} \end{aligned}$$

$$\begin{aligned} A_0 &\equiv A_{q\bar{q}}(\uparrow\uparrow)A_{t\bar{t}}(\uparrow\downarrow) = A_{q\bar{q}}(\uparrow\uparrow)A_{t\bar{t}}(\downarrow\uparrow) = -A_{q\bar{q}}(\downarrow\downarrow)A_{t\bar{t}}(\uparrow\downarrow) = -A_{q\bar{q}}(\downarrow\downarrow)A_{t\bar{t}}(\downarrow\uparrow) \\ &\propto 2k(\cos\psi\sin\theta - \sin\psi\cos\theta) = 2k\sin(\psi-\theta) \end{aligned}$$

Using the lepton flight directions described by (θ_+, φ_+) for ℓ^+ and (θ_-, φ_-) for ℓ^- in a basis, and the same calculation as the vector case can be applied.

$$\begin{aligned} 4\sigma \;\propto\; & (1-\cos\theta_+\cos\theta_-)\frac{|A_+|^2+|A_-|^2}{2} + (1+\cos\theta_+\cos\theta_-)|A_0|^2 \\ & -\sin\theta_+\cos\theta_-\mathrm{Re}\{e^{i\varphi_+}(A_+A_0^* - A_-^*A_0)\} \\ & -\cos\theta_+\sin\theta_-\mathrm{Re}\{e^{i\varphi_-}(A_+A_0^* - A_-^*A_0)\} \\ & -\sin\theta_+\sin\theta_-\{\mathrm{Re}(e^{i(\varphi_++\varphi_-)}A_+A_-^*) + \cos(\varphi_+-\varphi_-)|A_0|^2\} \end{aligned}$$

If we choose $\psi = \theta$ (**helicity basis**),

$$A_+ \propto 2k(1+\cos\theta)$$

$$A_- \propto -2k(1-\cos\theta)$$

$$A_0 = 0$$

$$4\sigma \propto (1-\cos\theta_+\cos\theta_-)\frac{A_+^2+A_-^2}{2} - \sin\theta_+\sin\theta_-\cos(\varphi_++\varphi_-)A_+A_-$$

$$\frac{A_+^2+A_-^2}{2} \propto 4k^2(1+\cos^2\theta) \qquad A_+A_- \propto 4k^2(1-\cos^2\theta) = 4k^2\sin^2\theta$$

Intuitive understanding for this result: For non-relastivistic approximation

$$\langle\gamma^\mu\rangle \sim (1; \boldsymbol{v}) \qquad \langle\gamma^\mu\gamma^5\rangle \sim (\boldsymbol{\sigma}\cdot\boldsymbol{v}; \boldsymbol{\sigma})$$

$q\bar{q} \to t\bar{t}$ via V-A current

$$\mathfrak{M}_{V-A} \propto \left\{\bar{v}_{\bar{q}}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)u_q\right\}\left\{\bar{u}_t\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)v_{\bar{t}}\right\} = M_{q\bar{q}}(\lambda_q\lambda_{\bar{q}})M_{t\bar{t}}(\lambda'_t\lambda'_{\bar{t}})$$

where $u_Q(\boldsymbol{p}_Q) = N_Q\begin{pmatrix} \chi_Q \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_Q}{E_Q+m_Q}\chi_Q \end{pmatrix}$ and $v_{\bar{Q}}(\boldsymbol{p}_{\bar{Q}}) = N_{\bar{Q}}\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_{\bar{Q}}}{E_{\bar{Q}}+m_{\bar{Q}}}\chi_{\bar{Q}} \\ \chi_{\bar{Q}} \end{pmatrix}$.

Assume $\boldsymbol{p} \equiv \boldsymbol{p}_q = -\boldsymbol{p}_{\bar{q}} = (0,0,p)$, $m_q = m_{\bar{q}} = 0$, and $E_q = E_{\bar{q}} = p$, then

$$
\begin{aligned}
M^{\mu}_{q\bar{q}} &= \bar{v}_{\bar{q}}\gamma^{\mu}\left(\frac{1-\gamma^5}{2}\right)u_q = \frac{1}{2}(J^{\mu}_{q\bar{q}} - A^{\mu}_{q\bar{q}}) \\
&\propto \frac{1}{2}\left[\left(0; 2\chi^{\dagger}_{\bar{q}}\sigma_1\chi_q, 2\chi^{\dagger}_{\bar{q}}\sigma_2\,\chi_q, 0\right) - \left\{0; 2\chi^{\dagger}_{\bar{q}}(-i\sigma_2)\chi_q, 2\chi^{\dagger}_{\bar{q}}(i\sigma_1)\,\chi_q, 0\right\}\right] \\
&= \left\{0; \chi^{\dagger}_{\bar{q}}(\sigma_1 + i\sigma_2)\chi_q, -i\chi^{\dagger}_{\bar{q}}(\sigma_1 + i\sigma_2)\chi_q, 0\right\}
\end{aligned}
$$

If we consider eigen states along z-axis for u_q and $v_{\bar{q}}$, i.e.

$$
\chi_q(\uparrow) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \chi_q(\downarrow) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \chi_{\bar{q}}(\uparrow) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \text{and}\ \chi_{\bar{q}}(\downarrow) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}
$$

and

$$
\sigma_1 + i\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
$$

then

$$
\begin{aligned}
&M^{\mu}_{q\bar{q}}(\uparrow\uparrow) \propto (0; 0, 0, 0) \\
&M^{\mu}_{q\bar{q}}(\uparrow\downarrow) \propto (0; 0, 0, 0) \\
&M^{\mu}_{q\bar{q}}(\downarrow\uparrow) \propto (0; 0, 0, 0) \\
&M^{\mu}_{q\bar{q}}(\downarrow\downarrow) \propto (0; -1, i, 0)
\end{aligned}
$$

Next, we consider $M_{t\bar{t}}$. In ZMF, i.e. $\boldsymbol{p}_t = -\boldsymbol{p}_{\bar{t}}$, $m_t = m_{\bar{t}}$, and $E_t = E_{\bar{t}}$

$$
\begin{aligned}
M^{\mu}_{t\bar{t}} &= \bar{u}_t\gamma^{\mu}\left(\frac{1-\gamma^5}{2}\right)v_{\bar{t}} = \frac{1}{2}(J^{\mu}_{t\bar{t}} - A^{\mu}_{t\bar{t}}) \\
&\propto \frac{1}{2}\left[-(1-\boldsymbol{\alpha}^2)\chi^{\dagger}_t\chi_{\bar{t}}\,; \chi^{\dagger}_t\{\sigma_i - (\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\sigma_i(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}) - (\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\sigma_i + \sigma_i(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\}\chi_{\bar{t}}\right]
\end{aligned},
$$

where $\boldsymbol{\alpha} = \dfrac{\boldsymbol{p}_t}{E_t + m_t} = k\begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$, $k^2 = \left(\dfrac{p_t}{E_t + m_t}\right)^2 = \dfrac{\gamma_t - 1}{\gamma_t + 1}$, then

$$
\begin{aligned}
M^{\mu}_{t\bar{t}} &= \frac{1}{2}(J^{\mu}_{t\bar{t}} - A^{\mu}_{t\bar{t}}) \\
&= \left(*; \frac{1}{2}\chi^{\dagger}_t\left\{\begin{bmatrix} (1 + k^2\mathrm{cos}2\theta)\sigma_1 - k^2\mathrm{sin}2\theta\cdot\sigma_3 \\ (1+k^2)\sigma_2 \\ (1 - k^2\mathrm{cos}2\theta)\sigma_3 - k^2\mathrm{sin}2\theta\cdot\sigma_1 \end{bmatrix} - \begin{bmatrix} 2i\,k\sigma_2\mathrm{cos}\theta \\ 2i\,k\sigma_3\mathrm{sin}\theta - 2i\,k\sigma_1\mathrm{cos}\theta \\ -2i\,k\sigma_2\mathrm{sin}\theta \end{bmatrix}\right\}\chi_{\bar{t}}\right)
\end{aligned}
$$

Suppose

$$
\chi_{t\uparrow} = \begin{pmatrix} \cos\frac{\psi}{2} \\ \sin\frac{\psi}{2} \end{pmatrix},\ \chi_{t\downarrow} = \begin{pmatrix} -\sin\frac{\psi}{2} \\ \cos\frac{\psi}{2} \end{pmatrix}\ \text{for top quark, and}
$$

$$
\chi_{\bar{t}\uparrow} = \begin{pmatrix} -\sin\frac{\psi}{2} \\ \cos\frac{\psi}{2} \end{pmatrix},\ \chi_{\bar{t}\downarrow} = \begin{pmatrix} -\cos\frac{\psi}{2} \\ -\sin\frac{\psi}{2} \end{pmatrix}\ \text{for anti-top quark,}
$$

then

$$
M^{\mu}_{t\bar{t}}(\uparrow\uparrow) \propto \frac{1}{2}\left(*; \begin{bmatrix} (1 + k^2\mathrm{cos}2\theta)\mathrm{cos}\psi + k^2\mathrm{sin}2\theta\mathrm{sin}\psi - 2k\,\mathrm{cos}\theta \\ -i(1+k^2) + 2i\,k\,\mathrm{sin}\psi\mathrm{sin}\theta + 2i\,k\,\mathrm{cos}\psi\mathrm{cos}\theta \\ * \end{bmatrix}\right)
$$

$$
M^{\mu}_{t\bar{t}}(\uparrow\downarrow) = M^{\mu}_{t\bar{t}}(\downarrow\uparrow) \propto \frac{1}{2}\left(*; \begin{bmatrix} -(1 + k^2\mathrm{cos}2\theta)\mathrm{sin}\psi + k^2\mathrm{sin}2\theta\mathrm{cos}\psi \\ 2i\,k\,\mathrm{cos}\psi\mathrm{sin}\theta - 2i\,k\,\mathrm{sin}\psi\mathrm{cos}\theta \\ * \end{bmatrix}\right)
$$

$$
M^{\mu}_{t\bar{t}}(\downarrow\downarrow) \propto \frac{1}{2}\left(*; \begin{bmatrix} -(1 + k^2\mathrm{cos}2\theta)\mathrm{cos}\psi - k^2\mathrm{sin}2\theta\mathrm{sin}\psi - 2k\,\mathrm{cos}\theta \\ -i(1+k^2) - 2i\,k\,\mathrm{sin}\psi\mathrm{sin}\theta - 2i\,k\,\mathrm{cos}\psi\mathrm{cos}\theta \\ * \end{bmatrix}\right)
$$

$$
\begin{aligned}
M_+ &\equiv M_{q\bar{q}}(\downarrow\downarrow)M_{t\bar{t}}(\downarrow\downarrow) \\
&\propto \frac{1}{2}\{(1+k^2\cos2\theta)\cos\psi + k^2\sin2\theta\sin\psi + 2k\cos\theta \\
&\qquad + (1+k^2) + 2k\sin\psi\sin\theta + 2k\cos\psi\cos\theta\} \\
M_- &\equiv M_{q\bar{q}}(\downarrow\downarrow)M_{t\bar{t}}(\uparrow\uparrow) \\
&\propto \frac{1}{2}\{-(1+k^2\cos2\theta)\cos\psi - k^2\sin2\theta\sin\psi + 2k\cos\theta \\
&\qquad + (1+k^2) - 2k\sin\psi\sin\theta - 2k\cos\psi\cos\theta\} \\
M_0 &\equiv M_{q\bar{q}}(\downarrow\downarrow)M_{t\bar{t}}(\uparrow\downarrow) = M_{q\bar{q}}(\downarrow\downarrow)M_{t\bar{t}}(\downarrow\uparrow) \\
&\propto \frac{1}{2}\{(1+k^2\cos2\theta)\sin\psi - k^2\sin2\theta\cos\psi - 2k\cos\psi\sin\theta + 2k\sin\psi\cos\theta\}
\end{aligned}
$$

Using the lepton flight directions described by $(\theta_+,\ \varphi_+)$ for ℓ^+ and $(\theta_-,\ \varphi_-)$ for ℓ^- in a basis, and the same calculation as the vector case can be applied.

$$
\begin{aligned}
|M_{q\bar{q}}(\downarrow\downarrow)M_{t\bar{t}}|^2 = \Big| &-e^{i(\varphi_+ + \varphi_-)/2}\cos\frac{\theta_+}{2}\sin\frac{\theta_-}{2}M_{q\bar{q}}(\downarrow\downarrow)M_{t\bar{t}}(\uparrow\uparrow) \\
&+e^{i(\varphi_+ - \varphi_-)/2}\cos\frac{\theta_+}{2}\cos\frac{\theta_-}{2}M_{q\bar{q}}(\downarrow\downarrow)M_{t\bar{t}}(\uparrow\downarrow) \\
&-e^{-i(\varphi_+ - \varphi_-)/2}\sin\frac{\theta_+}{2}\sin\frac{\theta_-}{2}M_{q\bar{q}}(\downarrow\downarrow)M_{t\bar{t}}(\downarrow\uparrow) \\
&+e^{-i(\varphi_+ + \varphi_-)/2}\sin\frac{\theta_+}{2}\cos\frac{\theta_-}{2}M_{q\bar{q}}(\downarrow\downarrow)M_{t\bar{t}}(\downarrow\downarrow)\Big|^2
\end{aligned}
$$

$$
\begin{aligned}
= \Big| &-e^{i(\varphi_+ + \varphi_-)/2}\cos\frac{\theta_+}{2}\sin\frac{\theta_-}{2}M_+ + e^{i(\varphi_+ - \varphi_-)/2}\cos\frac{\theta_+}{2}\cos\frac{\theta_-}{2}M_0 \\
&-e^{-i(\varphi_+ - \varphi_-)/2}\sin\frac{\theta_+}{2}\sin\frac{\theta_-}{2}M_0 + e^{-i(\varphi_+ + \varphi_-)/2}\sin\frac{\theta_+}{2}\cos\frac{\theta_-}{2}M_-\Big|^2
\end{aligned}
$$

$$
\begin{aligned}
= &\cos^2\frac{\theta_+}{2}\sin^2\frac{\theta_-}{2}|M_+|^2 + \cos^2\frac{\theta_+}{2}\cos^2\frac{\theta_-}{2}|M_0|^2 \\
&+\sin^2\frac{\theta_+}{2}\sin^2\frac{\theta_-}{2}|M_0|^2 + \sin^2\frac{\theta_+}{2}\cos^2\frac{\theta_-}{2}|M_-|^2 \\
&-2\cos^2\frac{\theta_+}{2}\sin\frac{\theta_-}{2}\cos\frac{\theta_-}{2}\mathrm{Re}(e^{i\varphi_-}M_+M_0^*) \\
&+2\cos\frac{\theta_+}{2}\sin\frac{\theta_+}{2}\sin^2\frac{\theta_-}{2}\mathrm{Re}(e^{i\varphi_+}M_+M_0^*) \\
&-2\cos\frac{\theta_+}{2}\sin\frac{\theta_+}{2}\sin\frac{\theta_-}{2}\cos\frac{\theta_-}{2}\mathrm{Re}\big(e^{i(\varphi_+ + \varphi_-)}M_+M_-^*\big) \\
&-2\cos(\varphi_+ - \varphi_-)\cos\frac{\theta_+}{2}\sin\frac{\theta_+}{2}\cos\frac{\theta_-}{2}\sin\frac{\theta_-}{2}|M_0|^2 \\
&+2\cos\frac{\theta_+}{2}\sin\frac{\theta_+}{2}\cos^2\frac{\theta_-}{2}\mathrm{Re}(e^{i\varphi_+}M_-^*M_0) \\
&-2\sin^2\frac{\theta_+}{2}\cos\frac{\theta_-}{2}\sin\frac{\theta_-}{2}\mathrm{Re}(e^{i\varphi_-}M_-^*M_0)
\end{aligned}
$$

$$
\begin{aligned}
4\sigma \propto &\frac{1}{4}(1+\cos\theta_+)(1-\cos\theta_-)|M_+|^2 + \frac{1}{4}(1-\cos\theta_+)(1+\cos\theta_-)|M_-|^2 \\
&+\frac{1}{2}(1+\cos\theta_+\cos\theta_-)|M_0|^2 \\
&-\frac{1}{2}\sin\theta_+\sin\theta_-\mathrm{Re}\big(e^{i(\varphi_+ + \varphi_-)}M_+M_-^*\big) \\
&-\frac{1}{2}\sin\theta_+\sin\theta_-\cos(\varphi_+ - \varphi_-)|M_0|^2 \\
&-\frac{1}{2}(1+\cos\theta_+)\sin\theta_-\mathrm{Re}(e^{i\varphi_-}M_+M_0^*) \\
&+\frac{1}{2}\sin\theta_+(1-\cos\theta_-)\mathrm{Re}(e^{i\varphi_+}M_+M_0^*) \\
&-\frac{1}{2}(1-\cos\theta_+)\sin\theta_-\mathrm{Re}(e^{i\varphi_-}M_-^*M_0) \\
&+\frac{1}{2}\sin\theta_+(1+\cos\theta_-)\mathrm{Re}(e^{i\varphi_+}M_-^*M_0)
\end{aligned}
$$

In case of real M_+, M_-, and M_0

$$
\begin{aligned}
4\sigma \;\propto\; & \frac{1}{4}(1+\cos\theta_+)(1-\cos\theta_-)M_+^2 + \frac{1}{4}(1-\cos\theta_+)(1+\cos\theta_-)M_-^2 \\
& + \frac{1}{2}(1+\cos\theta_+\cos\theta_-)M_0^2 \\
& - \frac{1}{2}\sin\theta_+\sin\theta_-\cos(\varphi_+ + \varphi_-)M_+M_- \\
& - \frac{1}{2}\sin\theta_+\sin\theta_-\cos(\varphi_+ - \varphi_-)M_0^2 \\
& - \frac{1}{2}(1+\cos\theta_+)\sin\theta_-\cos\varphi_- M_+M_0 \\
& + \frac{1}{2}\sin\theta_+(1-\cos\theta_-)\cos\varphi_+ M_+M_0 \\
& - \frac{1}{2}(1-\cos\theta_+)\sin\theta_-\cos\varphi_- M_-M_0 \\
& + \frac{1}{2}\sin\theta_+(1+\cos\theta_-)\cos\varphi_+ M_-M_0
\end{aligned}
$$

If we choose ψ which satisfies $M_0 = 0$, i.e.

$$(1+k^2\cos 2\theta)\sin\psi - k^2\sin 2\theta\cos\psi - 2k\cos\psi\sin\theta + 2k\sin\psi\cos\theta = 0$$

$$\sin\psi(1+k^2\cos 2\theta + 2k\cos\theta) = \cos\psi(k^2\sin 2\theta + 2k\sin\theta)$$

$$\tan\psi\{(1+k\cos\theta)^2 - k^2\sin^2\theta\} = 2k\sin\theta(1+k\cos\theta)$$

$$\tan\psi = \frac{2k\sin\theta(1+k\cos\theta)}{(1+k\cos\theta)^2 - k^2\sin^2\theta}$$

For ultra-relativistic $t\bar{t}$ production ($k \to 1$)

$$\tan\psi = \frac{2\sin\theta(1+\cos\theta)}{(1+\cos\theta)^2 - \sin^2\theta} = \frac{2\sin\theta(1+\cos\theta)}{2\cos\theta + 2\cos^2\theta} = \tan\theta$$

For threshold $t\bar{t}$ production ($k = 0$)

$$\tan\psi = 0$$

$$
\begin{aligned}
4\sigma \;\propto\; & \frac{1}{4}(1+\cos\theta_+)(1-\cos\theta_-)M_+^2 + \frac{1}{4}(1-\cos\theta_+)(1+\cos\theta_-)M_-^2 \\
& - \frac{1}{2}\sin\theta_+\sin\theta_-\cos(\varphi_+ + \varphi_-)M_+M_-
\end{aligned}
$$

$q\bar{q} \to t\bar{t}$ via V+A current

$$\mathfrak{M}_{V-A} \propto \left\{\bar{v}_{\bar{q}}\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u_q\right\}\left\{\bar{u}_t\gamma_\mu\left(\frac{1+\gamma^5}{2}\right)v_{\bar{t}}\right\} = L_{q\bar{q}}(\lambda_q\lambda_{\bar{q}})L_{t\bar{t}}(\lambda'_t\lambda'_{\bar{t}})$$

where $u_Q(\boldsymbol{p}_Q) = N_Q\begin{pmatrix} \chi_Q \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_Q}{E_Q+m_Q}\chi_Q \end{pmatrix}$ and $v_{\bar{Q}}(\boldsymbol{p}_{\bar{Q}}) = N_{\bar{Q}}\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_{\bar{Q}}}{E_{\bar{Q}}+m_{\bar{Q}}}\chi_{\bar{Q}} \\ \chi_{\bar{Q}} \end{pmatrix}$.

Assume $\boldsymbol{p} \equiv \boldsymbol{p}_q = -\boldsymbol{p}_{\bar{q}} = (0,0,p)$, $m_q = m_{\bar{q}} = 0$, and $E_q = E_{\bar{q}} = p$, then

$$
\begin{aligned}
L^\mu_{q\bar{q}} &= \bar{v}_{\bar{q}}\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u_q = \frac{1}{2}(J^\mu_{q\bar{q}} + A^\mu_{q\bar{q}}) \\
&\propto \frac{1}{2}\big[\big(0;2\chi^\dagger_{\bar{q}}\sigma_1\chi_q, 2\chi^\dagger_{\bar{q}}\sigma_2\,\chi_q, 0\big) + \big\{0;2\chi^\dagger_{\bar{q}}(-i\sigma_2)\chi_q, 2\chi^\dagger_{\bar{q}}(i\sigma_1)\,\chi_q, 0\big\}\big] \\
&= \big\{0;\chi^\dagger_{\bar{q}}(\sigma_1 - i\sigma_2)\chi_q, i\chi^\dagger_{\bar{q}}(\sigma_1 - i\sigma_2)\chi_q, 0\big\}
\end{aligned}
$$

If we consider eigen states along z-axis for u_q and $v_{\bar{q}}$, i.e.

$$\chi_q(\uparrow)=\begin{pmatrix}1\\0\end{pmatrix},\ \chi_q(\downarrow)=\begin{pmatrix}0\\1\end{pmatrix},\ \chi_{\bar{q}}(\uparrow)=\begin{pmatrix}0\\1\end{pmatrix},\ \text{and}\ \chi_{\bar{q}}(\downarrow)=\begin{pmatrix}-1\\0\end{pmatrix}$$

and

$$\sigma_1-i\sigma_2=\begin{pmatrix}0&1\\1&0\end{pmatrix}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}=2\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

then

$$L^\mu_{q\bar{q}}(\uparrow\uparrow)\propto(0;1,i,0)$$
$$L^\mu_{q\bar{q}}(\uparrow\downarrow)\propto(0;0,0,0)$$
$$L^\mu_{q\bar{q}}(\downarrow\uparrow)\propto(0;0,0,0)$$
$$L^\mu_{q\bar{q}}(\downarrow\downarrow)\propto(0;0,0,0)$$

Next, we consider $L_{t\bar{t}}$. In ZMF, i.e. $\boldsymbol{p}_t=-\boldsymbol{p}_{\bar{t}}$, $m_t=m_{\bar{t}}$, and $E_t=E_{\bar{t}}$

$$\begin{aligned}L^\mu_{t\bar{t}}&=\bar{u}_t\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)v_{\bar{t}}=\frac{1}{2}(J^\mu_{t\bar{t}}+A^\mu_{t\bar{t}})\\&\propto\frac{1}{2}\big[(1-\boldsymbol{\alpha}^2)\chi_t^\dagger\chi_{\bar{t}}\,;\,\chi_t^\dagger\{\sigma_i-(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\sigma_i(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})+(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\sigma_i-\sigma_i(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\}\chi_{\bar{t}}\big]\end{aligned},$$

where $\boldsymbol{\alpha}=\dfrac{\boldsymbol{p}_t}{E_t+m_t}=k\begin{pmatrix}\sin\theta\\0\\\cos\theta\end{pmatrix}$, $k^2=\left(\dfrac{p_t}{E_t+m_t}\right)^2=\dfrac{\gamma_t-1}{\gamma_t+1}$, then

$$\begin{aligned}L^\mu_{t\bar{t}}&=\frac{1}{2}(J^\mu_{t\bar{t}}+A^\mu_{t\bar{t}})\\&=\left(*;\frac{1}{2}\chi_t^\dagger\left\{\begin{bmatrix}(1+k^2\cos2\theta)\sigma_1-k^2\sin2\theta\cdot\sigma_3\\(1+k^2)\sigma_2\\(1-k^2\cos2\theta)\sigma_3-k^2\sin2\theta\cdot\sigma_1\end{bmatrix}+\begin{bmatrix}2i\,k\sigma_2\cos\theta\\2i\,k\sigma_3\sin\theta-2i\,k\sigma_1\cos\theta\\-2i\,k\sigma_2\sin\theta\end{bmatrix}\right\}\chi_{\bar{t}}\right)\end{aligned}$$

Suppose

$$\chi_{t\uparrow}=\begin{pmatrix}\cos\frac{\psi}{2}\\\sin\frac{\psi}{2}\end{pmatrix},\ \chi_{t\downarrow}=\begin{pmatrix}-\sin\frac{\psi}{2}\\\cos\frac{\psi}{2}\end{pmatrix}\ \text{for top quark, and}$$

$$\chi_{\bar{t}\uparrow}=\begin{pmatrix}-\sin\frac{\psi}{2}\\\cos\frac{\psi}{2}\end{pmatrix},\ \chi_{\bar{t}\downarrow}=\begin{pmatrix}-\cos\frac{\psi}{2}\\-\sin\frac{\psi}{2}\end{pmatrix}\ \text{for anti-top quark,}$$

then

$$L^\mu_{t\bar{t}}(\uparrow\uparrow)\propto\frac{1}{2}\left(*;\begin{bmatrix}(1+k^2\cos2\theta)\cos\psi+k^2\sin2\theta\sin\psi+2k\cos\theta\\-i(1+k^2)-2i\,k\sin\psi\sin\theta-2i\,k\cos\psi\cos\theta\\*\end{bmatrix}\right)$$

$$L^\mu_{t\bar{t}}(\uparrow\downarrow)=L^\mu_{t\bar{t}}(\downarrow\uparrow)\propto\frac{1}{2}\left(*;\begin{bmatrix}-(1+k^2\cos2\theta)\sin\psi+k^2\sin2\theta\cos\psi\\-2i\,k\cos\psi\sin\theta+2i\,k\sin\psi\cos\theta\\*\end{bmatrix}\right)$$

$$L^\mu_{t\bar{t}}(\downarrow\downarrow)\propto\frac{1}{2}\left(*;\begin{bmatrix}-(1+k^2\cos2\theta)\cos\psi-k^2\sin2\theta\sin\psi+2k\cos\theta\\-i(1+k^2)+2i\,k\sin\psi\sin\theta+2i\,k\cos\psi\cos\theta\\*\end{bmatrix}\right)$$

$$\begin{aligned}L_+&\equiv L_{q\bar{q}}(\uparrow\uparrow)L_{t\bar{t}}(\uparrow\uparrow)\\&\propto\frac{1}{2}\{(1+k^2\cos2\theta)\cos\psi+k^2\sin2\theta\sin\psi+2k\cos\theta\\&\qquad+(1+k^2)+2k\sin\psi\sin\theta+2k\cos\psi\cos\theta\}\end{aligned}$$

$$
\begin{aligned}
L_- &\equiv L_{q\bar{q}}(\uparrow\uparrow)L_{t\bar{t}}(\downarrow\downarrow) \\
&\propto \frac{1}{2}\{-(1+k^2\cos2\theta)\cos\psi - k^2\sin2\theta\sin\psi + 2k\cos\theta \\
&\qquad + (1+k^2) - 2k\sin\psi\sin\theta - 2k\cos\psi\cos\theta\}
\end{aligned}
$$

$$
\begin{aligned}
L_0 &\equiv L_{q\bar{q}}(\uparrow\uparrow)L_{t\bar{t}}(\uparrow\downarrow) = L_{q\bar{q}}(\downarrow\downarrow)L_{t\bar{t}}(\downarrow\uparrow) \\
&\propto \frac{1}{2}\{-(1+k^2\cos2\theta)\sin\psi + k^2\sin2\theta\cos\psi + 2k\cos\psi\sin\theta - 2k\sin\psi\cos\theta\}
\end{aligned}
$$

These results are almost same as V-A except for L_0 sign, i.e.

$$
L_\pm = M_\pm \qquad L_0 = -M_0.
$$

If we choose ψ which satisfies $L_0 = 0$, i.e.

$$
(1+k^2\cos2\theta)\sin\psi - k^2\sin2\theta\cos\psi - 2k\cos\psi\sin\theta + 2k\sin\psi\cos\theta = 0
$$

$$
\tan\psi = \frac{2k\sin\theta(1+k\cos\theta)}{(1+k\cos\theta)^2 - k^2\sin^2\theta}
$$

$$
\begin{aligned}
4\sigma &\propto \frac{1}{4}(1+\cos\theta_+)(1-\cos\theta_-)L_+^2 + \frac{1}{4}(1-\cos\theta_+)(1+\cos\theta_-)L_-^2 \\
&\quad - \frac{1}{2}\sin\theta_+\sin\theta_-\cos(\varphi_+ + \varphi_-)L_+L_-
\end{aligned}
$$

[EOF]