Off-diagonal basis in $q\bar{q} \to t\bar{t}$ process

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$$\xi = \sqrt{1-\beta^2}\tan\theta^*,$$

where ξ is defined as the clockwise angle of the off-diagonal basis w.r.t. the anti-top flight direction in the top rest frame.

Here we suppose

ψ: The angle of the off-diagonal basis w.r.t. the z-axis in $t\bar{t}$ c.m. frame (ZMF).

θ^*: The prodcution angle of the top quark w.r.t. the z-axis in ZMF.

Since $\xi + \psi = \theta^*$,

$$\tan(\theta^* - \psi) = \sqrt{1-\beta^2}\tan\theta^* = \frac{1}{\gamma}\tan\theta^*.$$

Apply the relation $\tan(\theta^* - \psi) = \dfrac{\tan\theta^* - \tan\psi}{1+\tan\theta^*\tan\psi}$

$$\frac{\tan\theta^* - \tan\psi}{1+\tan\theta^*\tan\psi} = \frac{1}{\gamma}\tan\theta^*$$

Therefore

$$\tan\psi = \frac{(\gamma-1)\tan\theta^*}{\gamma + \tan^2\theta^*}$$

$$= \frac{(\gamma-1)\sin\theta^*\cos\theta^*}{\gamma\cos^2\theta^* + \sin^2\theta^*} = \frac{(\gamma-1)\sin\theta^*\cos\theta^*}{\gamma - (\gamma-1)\sin^2\theta^*} = \frac{(\gamma^2-1)\sin\theta^*\cos\theta^*}{\gamma(\gamma+1) - (\gamma^2-1)\sin^2\theta^*}$$

Apply the relation $\beta^2\gamma^2 = \gamma^2 - 1$, then finally we obtain

$$\tan\psi = \frac{\beta^2\gamma^2\sin\theta^*\cos\theta^*}{\gamma(\gamma+1) - \beta^2\gamma^2\sin^2\theta^*} = \frac{\beta^2\sin\theta^*\cos\theta^*}{1 + 1/\gamma - \beta^2\sin^2\theta^*}.$$

Derivation of off-diagonal basis in $q\bar{q} \to t\bar{t}$ process

Consider $q\bar{q} \to g^*$ vertex,

$$\bar{u}_q\gamma^\mu v_{\bar{q}},$$

where $u_q(\boldsymbol{p}) = N_q\begin{pmatrix} \chi_q \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi_q \end{pmatrix}$ $\qquad v_{\bar{q}}(\boldsymbol{p}) = N_{\bar{q}}\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi_{\bar{q}} \\ \chi_{\bar{q}} \end{pmatrix}$.

In $q\bar{q}\,(t\bar{t})$ rest frame (ZMF),

$$\boldsymbol{p} \equiv \boldsymbol{p}_q = -\boldsymbol{p}_{\bar{q}} \qquad E \equiv E_q = E_{\bar{q}} \qquad m \equiv m_q = m_{\bar{q}},$$

then

$$\bar{u}_q\gamma^\mu v_{\bar{q}} = N_q N_{\bar{q}}\Big(0; -\chi_q^\dagger\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\sigma_i\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi_{\bar{q}} + \chi_q^\dagger\sigma_i\chi_{\bar{q}}\Big),$$

Suppose the case that $\boldsymbol{p} = (0,0,p)$, and $m = 0$,

$$\bar{u}_q\gamma^\mu v_{\bar{q}} = N_q N_{\bar{q}}\big(0; -\chi_q^\dagger\sigma_3\sigma_i\sigma_3\chi_{\bar{q}} + \chi_q^\dagger\sigma_i\chi_{\bar{q}}\big) = N_q N_{\bar{q}}\big(0; 2\chi_q^\dagger\sigma_1\chi_{\bar{q}}, 2\chi_q^\dagger\sigma_2\chi_{\bar{q}}, 0\big)$$

This indicates we have no longitudinal component for intermediate gluon polarization vector. This is corresponding to helicity conservation of initial partons.

Next consider $g^* \to t\bar{t}$ vertex. In ZMF,

$$\bar{u}_t\gamma^\mu v_{\bar{t}} = N_t N_{\bar{t}}\big(0;\chi_t^\dagger\{\sigma_i - (\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\,\sigma_i\,(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\}\chi_{\bar{t}}\big), \text{ where } \boldsymbol{\alpha}\equiv\frac{\boldsymbol{p}_t}{E_t+m_t}$$

Suppose the production angle θ of top quark w.r.t. z-axis,

$$\boldsymbol{\alpha} = k\begin{pmatrix}\sin\theta\\ 0\\ \cos\theta\end{pmatrix}, \text{ where } k\equiv\frac{p_t}{E_t+m_t} = \frac{\beta\gamma}{\gamma+1}$$

Therefore,

$$\boldsymbol{\sigma}\cdot\boldsymbol{\alpha} = k\,\sin\theta\cdot\sigma_1 + k\,\cos\theta\cdot\sigma_2\ ,$$

then

$$\sigma_i - (\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\,\sigma_i\,(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}) = \begin{bmatrix}(1+k^2\cos2\theta)\sigma_1 - k^2\sin2\theta\cdot\sigma_3\\ (1+k^2)\sigma_2\\ (1-k^2\cos2\theta)\sigma_3 - k^2\sin2\theta\cdot\sigma_1\end{bmatrix}$$

Consequently,

$$\bar{u}_t\gamma^\mu v_{\bar{t}} = N_t N_{\bar{t}}\left(0;\chi_t^\dagger\begin{bmatrix}(1+k^2\cos2\theta)\sigma_1 - k^2\sin2\theta\cdot\sigma_3\\ (1+k^2)\sigma_2\\ (1-k^2\cos2\theta)\sigma_3 - k^2\sin2\theta\cdot\sigma_1\end{bmatrix}\chi_{\bar{t}}\right)$$

Suppose an quantizing axis for top and anti-top spin, which have an angle ψ w.r.t. z-axis. Then eigenstates of two-component spinors for the quantization axis are written as follows:

$$\chi_{t\uparrow}(\psi) = \chi_{\bar{t}\downarrow} = \begin{pmatrix}\cos\frac{\psi}{2}\\ \sin\frac{\psi}{2}\end{pmatrix} \qquad \chi_{t\downarrow} = \chi_{\bar{t}\uparrow} = \begin{pmatrix}-\sin\frac{\psi}{2}\\ \cos\frac{\psi}{2}\end{pmatrix}.$$

Since z component of $\bar{u}_q\gamma^\mu v_{\bar{q}}$ is zero, we have only to take x and y components of $\bar{u}_t\gamma^\mu v_{\bar{t}}$ into account for the following 4 eigenstates of $t\bar{t}$:

$(\uparrow\uparrow)$ $\quad x:\ \chi_{t\uparrow}^\dagger[(1+k^2\cos2\theta)\sigma_1 - k^2\sin2\theta\cdot\sigma_3]\chi_{\bar{t}\uparrow} = (1+k^2\cos2\theta)\cos\psi + k^2\sin2\theta\,\sin\psi$

$\quad y:\ \chi_{t\uparrow}^\dagger[(1+k^2)\sigma_2]\chi_{\bar{t}\uparrow} = -i(1+k^2)$

$(\uparrow\downarrow)$ $\quad x:\ \chi_{t\uparrow}^\dagger[(1+k^2\cos2\theta)\sigma_1 - k^2\sin2\theta\cdot\sigma_3]\chi_{\bar{t}\downarrow} = (1+k^2\cos2\theta)\sin\psi - k^2\sin2\theta\,\cos\psi$

$\quad y:\ \chi_{t\uparrow}^\dagger[(1+k^2)\sigma_2]\chi_{\bar{t}\downarrow} = 0$

$(\downarrow\uparrow)$ $\quad x:\ \chi_{t\downarrow}^\dagger[(1+k^2\cos2\theta)\sigma_1 - k^2\sin2\theta\cdot\sigma_3]\chi_{\bar{t}\uparrow} = -(1+k^2\cos2\theta)\sin\psi + k^2\sin2\theta\,\cos\psi$

$\quad y:\ \chi_{t\downarrow}^\dagger[(1+k^2)\sigma_2]\chi_{\bar{t}\uparrow} = 0$

$(\downarrow\downarrow)$ $\quad x:\ \chi_{t\downarrow}^\dagger[(1+k^2\cos2\theta)\sigma_1 - k^2\sin2\theta\cdot\sigma_3]\chi_{\bar{t}\downarrow} = (1+k^2\cos2\theta)\cos\psi + k^2\sin2\theta\,\sin\psi$

$\quad y:\ \chi_{t\downarrow}^\dagger[(1+k^2)\sigma_2]\chi_{\bar{t}\downarrow} = i(1+k^2)$

Therefore, if we choose

$$(1+k^2\cos2\theta)\sin\psi - k^2\sin2\theta\,\cos\psi = 0\ ,$$

we obtain null component for $(\uparrow\downarrow)$ and $(\downarrow\uparrow)$. This condition could be rewritten as

$$\tan\psi = \frac{k^2\sin2\theta}{1+k^2\cos2\theta} = \frac{2(\gamma-1)\sin\theta\cos\theta}{\gamma+1+(\gamma-1)(1-2\sin^2\theta)} = \frac{(\gamma-1)\tan\theta}{\gamma+\tan^2\theta}\ ,$$

then we obtain

$$\frac{1}{\gamma}\tan\theta = \frac{\tan\theta-\tan\psi}{1+\tan\theta\tan\psi} = \tan\,(\theta-\psi)\ ,$$

Here we define ξ as the clockwise angle of the off-diagonal basis w.r.t. the anti-top flight direction in the $t\bar{t}$ rest frame, i.e. $\xi+\psi=\theta$, then

$$\tan\xi = \frac{1}{\gamma}\tan\theta = \sqrt{1-\beta^2}\tan\theta$$

Appendix

For $u_\uparrow(\psi) = v_\downarrow = \begin{pmatrix} \cos\frac{\psi}{2} \\ \sin\frac{\psi}{2} \end{pmatrix}, \qquad u_\downarrow = v_\uparrow = \begin{pmatrix} -\sin\frac{\psi}{2} \\ \cos\frac{\psi}{2} \end{pmatrix}$

$$\begin{pmatrix} u_\uparrow^\dagger\sigma_1 v_\uparrow & u_\uparrow^\dagger\sigma_2 v_\uparrow & u_\uparrow^\dagger\sigma_3 v_\uparrow \\ u_\uparrow^\dagger\sigma_1 v_\downarrow & u_\uparrow^\dagger\sigma_2 v_\downarrow & u_\uparrow^\dagger\sigma_3 v_\downarrow \\ u_\downarrow^\dagger\sigma_1 v_\uparrow & u_\downarrow^\dagger\sigma_2 v_\uparrow & u_\downarrow^\dagger\sigma_3 v_\uparrow \\ u_\downarrow^\dagger\sigma_1 v_\downarrow & u_\downarrow^\dagger\sigma_2 v_\downarrow & u_\downarrow^\dagger\sigma_3 v_\downarrow \end{pmatrix} = \begin{pmatrix} \cos\psi & -i & -\sin\psi \\ \sin\psi & 0 & \cos\psi \\ -\sin\psi & 0 & -\cos\psi \\ \cos\psi & i & -\sin\psi \end{pmatrix}$$