Probability Density of Asymmetry

Consider two distributions from Poisson distributions have μ_+ and μ_- as their mean values, respectively. Suppose we observe N_+ and N_- from the each distribution.

Poisson distribution:

$$\hat{P}(N|\mu) = \frac{\mu^N e^{-\mu}}{N!}$$

On the assumption of Bayesian with flat prior p.d.f, the p.d.f of μ when observe N is given by

$$\hat{P}(\mu|N) = \frac{\mu^N e^{-\mu}}{N!}$$

Probability density function of (μ_+, μ_-) when observe (N_+, N_-)

$$\tilde{P}(\mu_+, \mu_-|N_+, N_-) = \hat{P}(N_+|\mu_+)\hat{P}(N_-|\mu_-) = \frac{\mu_+^{N_+}\mu_-^{N_-}e^{-\mu_+}e^{-\mu_-}}{N_+!N_-!}$$

For (μ_+, μ_-), we define an asymmetry as:

$$A \equiv \frac{\mu_+ - \mu_-}{\mu_+ + \mu_-} \qquad (-1 < A < 1)$$

In the polar coordinates

$$\begin{aligned} \mu_+ &= r\cos\theta \\ \mu_- &= r\sin\theta \end{aligned} \qquad (0 \le r < \infty, 0 \le \theta \le \pi/2)$$

$$\tan\theta = \frac{\mu_-}{\mu_+} = \frac{1-A}{1+A}$$

$$d\theta = \frac{d\theta}{dA}dA = \frac{-2\cos^2\theta}{(1+A)^2}dA$$

$$\tilde{P}(\mu_+, \mu_-) = \tilde{P}(r, \theta) = \frac{r^{N_+ + N_-}\cos^{N_+}\theta\,\sin^{N_-}\theta\,e^{-r(\cos\theta+\sin\theta)}}{N_+!N_-!}$$

$$dP = \int_{r=0}^{\infty}\tilde{P}(r,\theta)dr\cdot|r\,d\theta| = \int_{r=0}^{\infty}\tilde{P}(r,\theta)dr\cdot r\left|\frac{d\theta}{dA}\right|dA$$

$$\begin{aligned} \frac{dP}{dA} &= \int_0^{\infty} dr\cdot r\tilde{P}(r,\theta)\frac{d\theta}{dA} \\ &= \int_0^{\infty} dr\cdot r\,\frac{r^{N_+ + N_-}\cos^{N_+}\theta\,\sin^{N_-}\theta\,e^{-r(\cos\theta+\sin\theta)}}{N_+!N_-!}\,\frac{2\cos^2\theta}{(1+A)^2} \\ &= 2\,\frac{\cos^{N_+}\theta\,\sin^{N_-}\theta}{N_+!N_-!}\,\frac{\cos^2\theta}{(1+A)^2}\int_0^{\infty} dr\cdot r^{N_+ + N_- + 1}\,e^{-r(\cos\theta+\sin\theta)} \end{aligned}$$

From well known Laplace transformation

$$\int_0^{\infty} dr\cdot r^{\nu}\,e^{-rs} = \frac{\Gamma(\nu+1)}{s^{n+1}} \qquad (-1 < \nu < \infty)$$

$$\begin{aligned} \frac{dP}{dA} &= 2\,\frac{\cos^{N_+}\theta\,\sin^{N_-}\theta}{N_+!N_-!}\,\frac{\cos^2\theta}{(1+A)^2}\,\frac{(N_+ + N_- + 1)!}{(\cos\theta+\sin\theta)^{N_+ + N_- + 2}} \\ &= 2\,\frac{(N_+ + N_- + 1)!}{N_+!N_-!}\,\frac{1}{(1+A)^2}\,\frac{\tan^{N_-}\theta}{(1+\tan\theta)^{N_+ + N_- + 2}} \\ &= 2\,\frac{(N_+ + N_- + 1)!}{N_+!N_-!}\,\frac{1}{(1+A)^2}\left(\frac{1-A}{1+A}\right)^{N_-}\left(\frac{1+A}{2}\right)^{N_+ + N_- + 2} \\ &= \frac{1}{2}\,\frac{(N_+ + N_- + 1)!}{N_+!N_-!}\left(\frac{1-A}{2}\right)^{N_-}\left(\frac{1+A}{2}\right)^{N_+} \end{aligned}$$

[EOF]